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Operator Algebra in Chern-Simons Theory on a Torus

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Abstract

We consider Chern-Simons gauge theory on a torus with both nonrelativistic and relativistic matter. It is shown that the Hamiltonian and two total momenta commute among themselves only in the physical Hilbert space. We also discuss relations among degenerate physical states, degenerate vacua, and the existence of multicomponent Schrödinger wavefunctions.

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Chern-Simons field theory with matter coupling have attracted intense interest in recent years, owing to its relevance to condensed matter systems such as quantum Hall systems, and possibly high T_c superconductors [1]. Setting aside such physical applicability, Chern-Simons theory is in itself very interesting in view of its rich and beautiful mathematical structures. As such, various aspects of this theory deserve a careful study. Much of recent efforts have been directed towards the issue of consistent quantisation of the theory [2,3].

While the majority of works in the field is concerned with planar systems, Chern-Simons field theory on compact Riemann surfaces has also captured considerable interests [4–14]. It has even richer structures, which, being topological in nature, are absent in planar system. Among them are the multicomponent structure of many-body wavefunctions [9,11,14] and the degeneracy of physical states [5]. Moreover, the analysis on a torus is mathematically rigorous, being free from infrared divergences and ambiguity in boundary conditions at space infinity on a plane.

We extend our previous analysis [14], examining algebraic relations among various operators, especially the Hamiltonian H and total momenta P^k, with an eye on whether translation invariance is maintained or broken in the presence of matter. We shall also discuss a possible link between degeneracy of physical states and the multicomponent structure of wavefunctions.

It has been argued by Chen et al.[15] that the microscopic translation invariance of the anyon superconductivity model is broken in the mean field approximation, and is restored in the random phase approximation thanks to the presence of the phonon mode. Our consideration is at the microscopic level. We shall show that H and P^k do not commute with each other as operators, whereas they do commute in the physical Hilbert space. Therefore, if an effective theory is formulated in terms of physical excitations, the translation invariance must be maintained manifestly in each mode. Similar arguments have been given by Iengo, Lechner, and Li [13] on a torus, and by Banerjee [16] on a plane. Our argument, however, differs from theirs in detail.

We consider two models, Chern-Simons gauge theories on a torus with a non-relativistic matter field and with a relativistic Dirac field. We shall find algebraic relations universal in both theories.

We first analyse the nonrelativistic case. The Lagrangian is given by

$$\mathcal{L} = \frac{\kappa}{4\pi} \, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \mathcal{L}_{\rm matter} \tag{1}$$

where

$$\mathcal{L}_{\rm matter} = \frac{i}{2}\Big\{\psi^\dagger D_0 \psi - (D_0\psi)^\dagger \psi\Big\} - \frac{1}{2m}(D_k\psi)^\dagger (D_k\psi) \quad , \tag{2}$$

$D_0 = \partial_0 + i a_0$, and $D_k = \partial_k - i a^k$. Consistency of the theory requires that the coefficient of the Chern-Simons term κ be a fractional ratio, $\kappa = N/M$, where N and M are coprime integers [7,8]. The funamental domain of the torus is given by $0 \le x_j \le L_j, j = 1, 2$. Boundary conditions of the fields are then [4,14]

$$\begin{aligned} a_\mu[T_j(x)] &= a_\mu[x] + \partial_\mu \beta_j(x) \ , \\ \psi[T_j(x)] &= e^{-i\beta_j(x)} \ \psi(x) \ , \end{aligned} \tag{3}$$

where $T_j : \ x_j \rightarrow x_j + L_j$ $(j = 1, 2)$. That is, the fields return to their original values up to gauge transformations after translations along non-contractible loops. The requirement of smoothness of the field operator $\psi(x)$ in the covering space, $\psi[T_1 \cdot T_2(x)] = \psi[T_2 \cdot T_1(x)]$, leads to quantisation of the Chern-Simons flux, $\Phi = \int d\mathbf{x} \ f_{12} = 2\pi m$ ($m :$ integers), and a constraint on the β_j's: $\big\{\beta_1(T_2 x) - \beta_1(x)\big\} - \big\{\beta_2(T_1 x) - \beta_2(x)\big\} = -2\pi m$. Typical β_j's which solve this constraint are $\beta_j(x) = -\epsilon^{jk} \pi m x_k / L_k$, which will be taken in the rest of the paper.

Canonical energy-momentum tensors, $T_c^{\mu\nu}$, derived from (1) and (2) are not gauge-invariant, and therefore are not well defined on a torus in view of the boundary conditions (3). The gauge-invariant energy-momentum tensors $T_I^{\mu\nu}$ are obtained by adding the term $(\kappa/4\pi)\,\partial_\rho\,(\epsilon^{\rho\mu\sigma}a_\sigma a^\nu)$ to $T_c^{\mu\nu}$ and making use of the equations of motion [3]:

$$
\begin{aligned}
T_I^{\mu\nu} &= \begin{bmatrix} \dfrac{i}{2}\{\psi^\dagger D^\nu\psi - (D^\nu\psi)^\dagger\psi\} \\ -\dfrac{1}{2m}\{(D^k\psi)^\dagger D^\nu\psi + (D^\nu\psi)^\dagger D^k\psi\} \end{bmatrix} \\
&\quad - g^{\mu\nu}\left\{\frac{i}{2}\Big(\psi^\dagger D^0\psi - (D^0\psi)^\dagger\psi\Big) - \frac{1}{2m}(D^k\psi)^\dagger D^k\psi\right\} \quad .
\end{aligned}
\tag{4}
$$

where the upper (lower) entries in the square bracket give the $\mu = 0$ ($\mu = k$) component of $T_I^{\mu\nu}$. From (4) we obtain the Hamiltonian and total momentum operators

$$
\begin{aligned}
H &= \frac{1}{2m}\int d\mathbf{x}\ (D_k\psi)^\dagger(D_k\psi) \quad , \\
P^k &= -i\int d\mathbf{x}\ \psi^\dagger D_k\psi \quad .
\end{aligned}
\tag{5}
$$

To quantize the theory we note that the Chern-Simons field equation $(\kappa/4\pi)\varepsilon^{\mu\nu\rho}f_{\nu\rho}= j^\mu$ implies that Chern-Simons fields $a_\mu(x)$ are determined by the matter field except for non-integrable phases of Wilson line integrals along non-contractible loops on a torus. Solving the field equations in the radiation gauge $div\,\mathbf{a} = 0$, one finds [4]

$$
\begin{aligned}
a^j(x) &= \frac{\theta_j(t)}{L_j} - \frac{\Phi}{2L_1L_2}\,\epsilon^{jk}x_k + \hat{a}^j(x) \quad , \\
\hat{a}^j(x) &= \frac{2\pi}{\kappa}\int d\mathbf{y}\ \epsilon^{jk}\nabla_k^x G(\mathbf{x}-\mathbf{y})\ \Big(j^0(y) + \frac{\kappa\Phi}{2\pi L_1L_2}\Big) \quad , \\
a_0(x) &= -\frac{2\pi}{\kappa}\int d\mathbf{y}\ G(\mathbf{x}-\mathbf{y})\ (\partial_1 j^2 - \partial_2 j^1)(y) \quad ,
\end{aligned}
\tag{6}
$$

where $j^0 = \psi^\dagger\psi$, $j^k = -(i/2m)\big\{\psi^\dagger D_k\psi - (D_k\psi)^\dagger\psi\big\}$, and $G(\mathbf{r})$ is the periodic Green's function on a torus, satisfying $\Delta G(\mathbf{r}) = \delta(\mathbf{r}) - (1/L_1L_2)$.

The constant parts of θ_j are non-integrable phases of the Wilson line integrals. Residual gauge transformations that respect the boundary conditions (3) are given by a gauge function $\Lambda(x) = -\big(\sum_j 2\pi n_j x_j/L_j\big) + \tilde\Lambda(x)$ where n_j's are integers and $\tilde\Lambda(x)$ is periodic. The degree $\tilde\Lambda(x)$ has been made use of to obtain (6), whereas the rest of $\Lambda(x)$ constitutes large gauge transformations inducing $\theta_j \to \theta_j + 2\pi n_j$. The invariance under large gauge transformations must be imposed on physical quantities.

In quantum theory, physical degrees of freedom are the matter fields ψ (taken to be fermionic), and the non-integrable phases θ_j's. ψ's satify $\{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})$ in the fundamental domain of the torus. θ_1 and θ_2 are canonical conjugate of each other([4,5]), $[\theta_1, \theta_2] = 2\pi i/\kappa$, as the Lagrangian contains $(\kappa/4\pi)(\theta_2\dot\theta_1 - \theta_1\dot\theta_2)$. In writing the Hamiltonian (5) in terms of these variables with the aid of (6), there arises an ordering ambiguity. We shall take the ordering of ψ and $\psi^\dagger$ as it is in (5).

The equation of motion for θ_j is

$$\dot\theta_j = \frac{1}{i}\,[\theta_j, H] = -\epsilon^{jk}\,\frac{2\pi}{\kappa L_k}\,J^k \ , \tag{7}$$

where $J^k \equiv \int d\mathbf{x}\ j^k$. For $\psi(x)$

$$i\dot\psi(x) = \Big\{ -\frac{1}{2m} D_k^2 + a_0(x) + g(x)\Big\}\psi(x) \ , \tag{8}$$

where $g(x) = (1/2m)(2\pi/\kappa)^2 \int d\mathbf{y}\ [\nabla_k G(\mathbf{x}-\mathbf{y})]^2 \psi^\dagger\psi(y)$. Eq. (8) differs from the classical equation by the $g(x)$ term [14]. (See also ref. [3] for an analogous result on a plane.)

It is important to recognize that the expression (6) is not completely equivalent to the Chern-Simons field equations $(\kappa/4\pi)\varepsilon^{\mu\nu\rho} f_{\nu\rho} = j^\mu$. Insertion of (6) into a_μ's in the equations yields two nontrivial relations, one Eq. (7) and the other

$$Q + \frac{\kappa}{2\pi}\Phi \approx 0 \quad , \tag{9}$$

where $Q = \int d\mathbf{x}\ j^0$. Φ is the flux fixed by the boundary conditions (3) with the given $\beta_j(x)$'s, and Q is conserved as a consequence of Eq. (8). Since the relation (9) does not follow from the Hamiltonian and commutation relations, it has to be imposed as a constraint. We have adopted the notation $\approx$ to signify this.

The quantum field theory defined by (6) – (9) with the specified ordering of operators is precisely equivalent to quantum mechanics of anyons [3,14]. This is the reason for our analysing Chern-Simons gauge theory in the present form.

We now compute commutators of P^k and H. Note that since $\int d\mathbf{x}\ \psi^\dagger \hat{a}^k \psi = 0$, $P^k = -i \int d\mathbf{x}\ \psi^\dagger D_k \psi = -i \int d\mathbf{x}\ \psi^\dagger \overline{D}_k \psi$, where $\overline{D}_k \equiv \partial_k - i(\theta_k/L_k) + i(\epsilon^{kl} x_l \Phi/2L_1L_2)$. It follows that

$$[P^k, \psi(x)] = i\overline{D}_k \psi(x) \quad , \quad [P^k, \theta_j\,] = -\epsilon^{kj}\,\frac{2\pi i}{\kappa L_k} Q \quad . \tag{10}$$

In particular, the change in a gauge-invariant operator generated by P^k is a total derivative. For instance, $[P^k, \psi^\dagger\psi(x)] = i\partial_k\{\psi^\dagger\psi(x)\}$. With the aid of (10) commutators among the operators P^k and H are found to be

$$\begin{aligned} [P^j, P^k] &= i\epsilon^{jk} \frac{2\pi}{\kappa L_1 L_2}\, Q\, \left(Q + \frac{\kappa}{2\pi}\Phi\right) \quad , \\ [P^j,\, H\,] &= i\epsilon^{jk} \frac{2\pi}{\kappa L_1 L_2}\, J^k \left(Q + \frac{\kappa}{2\pi}\Phi\right) \quad , \end{aligned} \tag{11}$$

Note that $J^k = P^k/m$ in nonrelativisitc theory.

P^k's and H commute among themselves only up to the constraint (9). Hence in the physical Hilbert space these operators commute, and translation invariance is maintained. Our conclusion differs from Iengo et al.'s claim [13] that H and P^k commute among themselves as operators. Banerjee's analysis on a plane [16] is in conformity with ours, although a different gauge is chosen.

There are two other sets of important operators, Wilson line operators $W_j = e^{i\theta_j}$ and generators of large gauge transformation $U_j = \exp\left\{i\epsilon^{jk}\kappa\,\theta_k - 2\pi i \int d\mathbf{x}\ (x_j/L_j)\,\psi^\dagger\psi(x)\right\}$ (U_j is well defined. If the coefficient of the integral were a fraction of $2\pi i$, there would arise inconsistency in $U_j\psi(x)U_j^{-1}$ combined with (3).)

U_j and W_j satisfy dual relations $W_1 W_2 = e^{-2\pi i/\kappa}\, W_2 W_1$, $U_1 U_2 = e^{-2\pi i\kappa}\, U_2 U_1$, and $[W_k, U_j] = 0$. Also $[U_j, P^k]$ =$[U_j, H] = 0$, as P^k and H are gauge-invariant. Commutator relations among W_j, P^k and H are, however, non-trivial:

$$\begin{aligned} [W_j, P^k] &= \epsilon^{jk} \frac{2\pi}{\kappa L_k}\, Q\, W_j \quad , \\ [W_j,\, H\,] &= \epsilon^{jk} \frac{\pi}{\kappa L_k} (J^k W_j + W_j J^k) \quad , \end{aligned} \tag{12}$$

In the nonrelativistic theory $J^k = P^k/m$ so that W_j's map an eigenstate into another eigenstate corresponding to different momemta and energy.

Most of the above results can be directly carried over to Chern-Simons gauge theory coupled to a Dirac field. In place of (2) we have $\mathcal{L}_{\rm matter} = \frac{i}{2}\{\overline{\psi}\gamma^\mu D_\mu\psi - (\overline{D_\mu\psi})\gamma^\mu\psi\} - m\overline{\psi}\psi$. The current is given by $j^\mu = \overline{\psi}\gamma^\mu\psi$. Gauge-invariant energy-momentum tensors are given by $T_I^{\mu\nu} = \frac{i}{2}\{\overline{\psi}\gamma^\mu D^\nu\psi - (\overline{D^\nu\psi})\gamma^\mu\psi\}$. It follows that

$$\begin{aligned} H &= \int d\mathbf{x}\; \overline{\psi}\,(-i\gamma^k D_k + m)\psi \quad , \\ P^k &= -i \int d\mathbf{x}\; \psi^\dagger D_k \psi \quad . \end{aligned} \tag{13}$$

Note that in the Dirac case J^k and P^k are independent quantities.

Most of the relations obtained for the nonrelativisitc case remain valid for the Dirac case with the substitution $j^\mu = \overline{\psi}\gamma^\mu\psi$ being made. The only change to be made is the equation for ψ: $i\dot{\psi} = \gamma^0(-i\gamma^k D_k + m + a_0\gamma^0)\psi(x)$. The relation (6) and the constraint (9) remain intact. Direct computations confirm (7), (10), and particularly the fundamental algebraic relations (11) and (12). J^k is not conserved even in the physical Hilbert space, however. Therefore W_j no longer maps an eigenstate of H into another.

We stress that the relation (11) and (12) are universal. They are independent of details of theories.

We now return to the non-relativistic theory and consider the representation of P^k and H in the corresponding quantum-mechanical anyon system. The case of an integer $\kappa = N$ has been analysed in ref. [14]. There are N degenerate vacua $|0_a\rangle$ $(a = 0, \cdots, N-1)$. q-body Schrödinger wavefunctions are given by $\phi_a(\mathbf{x}_1, \ldots, \mathbf{x}_q; t) = (q!)^{-\frac{1}{2}} \langle 0_a | \, \Omega \, \psi(x_1) \cdots \psi(x_q) | \Psi_q \rangle$ where $\Omega = \exp\big\{ -i \sum_{j=1}^q [(x_1^{(j)} \theta_1 / L_1) + (x_2^{(j)} \theta_2 / L_2)] \big\}$. The operator Ω is necessary to insure invariance under large gauge transformations. Wavefunctions must have N components as a consequence of the vaccum degeneracy. They realize the braid group algebra on a torus.

The representation of P^k, $\hat{P}^k \, \phi_a \equiv (q!)^{-1/2} \langle 0_a | \Omega \psi(x_1) \ldots \psi(x_q) P^k | \Psi_q \rangle$, is found by permuting P^k to the left of the ψ's and Ω. The result is simple:

$$\hat{P}^k = -i \sum_{j=1}^q \nabla_k^{(j)} \ . \tag{14}$$

The Hamiltonian, $\hat{H}$, is [14]:

$$\begin{aligned} \hat{H} &= -\frac{1}{2m} \sum_{j=1}^q \big[\nabla_k^{(j)} - i \tilde{a}_{\rm f}^k(\mathbf{r}_j) \big]^2 \ , \\ \tilde{a}_{\rm f}^k(\mathbf{r}_j) &= \frac{2\pi}{\kappa} \, \epsilon^{kl} \sum_{p \neq j} \Big\{ \frac{1}{2L_1 L_2} \, (x_l^{(j)} - x_l^{(p)}) + \partial_l^{(j)} G(\mathbf{r}_j - \mathbf{r}_p) \Big\} \ . \end{aligned} \tag{15}$$

It is obvious that all $\hat{P}^k$ and $\hat{H}$ commute with each other. The same conclusion has been reached by Iengo et al. [13] in a different formulation in which the non-integrable phases appear explicitly in the expressions of $\hat{P}^k$ and $\hat{H}$. Our expressions (14) and (15) do not contain the θ_j's.

For the general case where κ is fractional, $\kappa = N/M$ (N, M coprime), it is known that there are NM degenerate vacua [7,8]. At present there are two approaches of interpreting the nature of these vacua.

Following Polychronakos [8], one might consider, of the NM possible vacua, only N distinct physical vacua, each having M gauge copies as generated by U_j. Hence one considers a fixed combination of these gauge copies to represent a physical vacuum. This can be seen as a sort of "gauge fixing". Adopting the same procedure to our case, we will still have a multicomponent wavefunction, but now $|\Psi_q\rangle$ and $|0_a\rangle$ ($a = 0, \ldots, N-1$) are linear combinations of M different gauge-equivalent physical states and vacua, respectively.

The situation is different, however, if we do not regard U_j as operators of gauge transformation, but instead as physical operators generated by some physical tunnelling processes. Such consideration is particularly appropriate when the model defined by (1) and (2) represents an effective theory of, for instance, fractional quantum Hall effect (FQHE) (with external magnetic fields added). This leads to a conclusion that physical states must be degenerate as emphasized by Wen and Niu [5]. What we have seen here is that, not only must physical states be degenerate, but their wavefunctions must also have multiple components.

To be precise, let us denote the NM degenerate vacua by $|0_{ab}\rangle$ ($a = 0, \ldots, N-1; b = 0, \ldots, M-1$). Noting that U_1^M and U_2^M commute with each other, we choose them to be eigenstates of U_i^M: $U_i^M|0_{ab}\rangle = e^{i\lambda_i}|0_{ab}\rangle$ ($j = 1, 2$). Then in the θ_1-representation one finds [17] that $u_{ab}(\theta_1) = \langle\theta_1|0_{ab}\rangle = e^{ib(\lambda_2+N\theta_1)/M + i\lambda_1\theta_1/2\pi M}\ \delta_{2\pi}[\theta_1 + (\lambda_2 - 2\pi Ma)/N]$. Actions of the U_i, W_j on $|0_{ab}\rangle$ are :

$$\begin{aligned} U_1|0_{ab}\rangle &= e^{i(\lambda_1+2\pi Nb)/M}|0_{ab}\rangle \quad , \qquad & W_1|0_{ab}\rangle &= e^{-i(\lambda_2-2\pi Ma)/N}|0_{ab}\rangle \quad , \\ U_2|0_{ab}\rangle &= e^{i\lambda_2/M}|0_{a,b-1}\rangle \quad , \qquad & W_2|0_{ab}\rangle &= e^{i\lambda_1/N}|0_{a-1,b}\rangle \quad . \end{aligned} \tag{16}$$

We require the physical states $|\Psi\rangle$ to be invariant under U_i^{-M} as well, $U_i^{-M}|\Psi\rangle = e^{i\omega_i}|\Psi\rangle$. Since U_1 and U_2 commute with U_1^M and U_2^M, $U_i^{-1}|\Psi\rangle$ is also a physical state with the same eigenvalues. But as U_1 and U_2 do not commute, the states generated by them must be degenerate. It follows from $U_1U_2 = e^{-2\pi iN/M}\,U_2U_1$ that M degenerate states $|\Psi^k\rangle$ ($k = 0, \ldots, M-1$) satisfy [5]

$$U_1^{-1}|\Psi^k\rangle = e^{i\nu_k^1}|\Psi^k\rangle \quad , \quad U_2^{-1}|\Psi^k\rangle = e^{i\nu_k^2}|\Psi^{k+1}\rangle \quad . \tag{17}$$

In general, a q-particle state $|\Psi_q^k\rangle$ can be constructed from the vacua $|0_{ab}\rangle$ satisfying (16) by

$$|\Psi_q^k\rangle = \frac{1}{\sqrt{q!}} \sum_{a,b} \int d\mathbf{x}_1 \cdots d\mathbf{x}_q \ \phi_{ab}^k(\mathbf{x}_1, \ldots, \mathbf{x}_q; t) \psi^\dagger(x_q) \cdots \psi^\dagger(x_1) \Omega^\dagger |0_{ab}\rangle / \langle 0_{ab}|0_{ab}\rangle \quad , \quad (18)$$

where $\phi_{ab}^k(\mathbf{x}_1, \ldots, \mathbf{x}_q; t) = (q!)^{-\frac{1}{2}} \langle 0_{ab}|\Omega\psi(x_1) \ldots \psi(x_q)|\Psi_q^k\rangle$. To satisfy (17), however, $|\Psi_q^k\rangle$ can only involve $|0_{ak}\rangle$, *i.e.* $b = k$, since $|0_{ab}\rangle$ pick up diferent phases for different values of b under the action of U_1^{-1}. Furthermore, (i) $\nu_k^1 = -(\lambda_1 + 2\pi N k)/M$, $\nu_k^2 = -\lambda_2/M$; (ii) $\phi_{ab}^k = 0$ for $b \neq k$ and $\phi_{a,k+1}^{k+1} = \phi_{ak}^k$; and (iii) $\omega_j = -\lambda_j$. So states are M-fold degenerate, and their wavefunctions take the form of $(N \times M)$-component matrices ϕ_{ab}^k with non-vanishing entries only in the $k^{\rm th}$ column.

Particularly, in the $\kappa = 1/M$ case, which is of relevance to FQHE, elememtary particles have statistics $\theta_s = -M\pi$ and therefore they are either bosons or fermions depending on whether M is odd or even. Many-body states are nevertheless M-fold degenerate, and their wavefunctions have M components. Implications of these degenerate multicomponent wavefunctions in the braid group structure of quasi-particles have yet to be studied. In any case, independent of the two approaches just discussed, $\hat{P}^k$ and $\hat{H}$ are represented by (14) and (15), respectively.

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